Developing relational thinking: The importance of the equals sign

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Developing relational thinking: The importance of the equals sign

- Researchers have highlighted the importance of students developing relational thinking rather than relying on calculations to solve mathematical tasks.

- Relational thinking involves students recognizing and understanding the relationship between given quantities.

- To use relational thinking, students need to understand the properties of the operations and recognise when these rules can be used e.g. addition and multiplication are commutative but subtraction and division are not.

- In this session, the focus will be on materials and tasks that teachers can use to encourage students to move beyond the belief that the equals sign means ‘give an answer’ so that they recognise when relational thinking can be used.

Your thoughts:

Why do you think many students struggle with equality?

The meaning of the equals sign

Many students do not understand the mathematical meaning of the equals sign: that the expressions on either side have the same value. Instead, they believe that an equals sign indicates where to write an answer.

Solving an open number sentence

http://smartvic.com/teacher/mdc/structure/St25001P.html

The teacher wrote an open number sentence:

$$7 + 6 = \square + 5$$

and asked children to find the missing number and to say how they found the missing number

There are four different responses:

Luke

Luke wrote 7 + 6 = **13** + 5

Teacher: Luke, what number did you put in the box?

Luke: Thirteen

Teacher: How did you decide?

Luke: 7 and 6 are 13

Teacher: What about the 5?

Luke: It doesn't matter. The answer to 7 + 6 is 13

Teacher: What is the 5 doing then?

Luke: It's just there.

Cameron

Cameron wrote 7 + 6 = 18 + 5

Teacher: Cameron, what number did you put in the box?

Cameron: Eighteen

Teacher: How did you decide?

Cameron: 7 and 6 are 13 and 5 more is 18

Teacher: Does 7 plus 6 equal to 18 plus 5?

Cameron: 7 + 6 is 13 and 5 more is 18

Fiona

Fiona wrote 7 + 6 = **8** + 5

Teacher: Fiona, what number did you put in the box?

Fiona: Eight

Teacher: How did you decide?

Fiona: 7 and 6 gives 13 and I then thought what number goes with 5 to give 13.

7 + 6 is 13 and 5 + 8 is 13

Chris

Chris wrote 7 + 6 = **8** + 5

Teacher: Chris, what number did you put in the box?

Chris: Eight

Teacher: How did you decide?

Chris: (Points to the numbers)

$$7 + 6 = \square + 5$$

5 is one less than 6, so you need a number that is one more than 7 to go in the □ so it all balances.

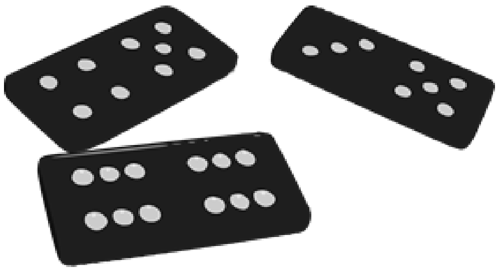
Important properties

1. Commutativity: a + b = b + a

4 + 5 = 5 + 4   3 + 5 = 5 + 3

But a – b is not the same as b – a

12 – 6 is not equal to 6 - 12



Addition is commutative but subtraction is not commutative.

2. Additive Identity Law

a + 0 = a   e.g. 6 + 0 = 6

a – 0 = a   e.g. 6 – 0 = 6

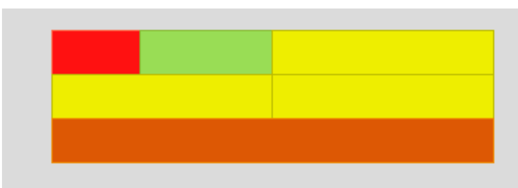
See https://www.youtube.com/watch?v=xaZyveOzdSw

Associative Law

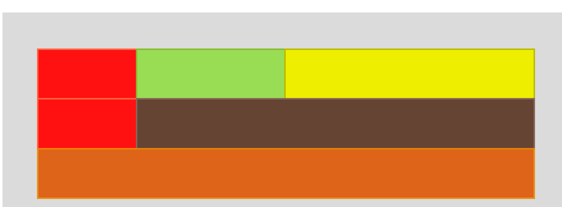
$(a + b) + c = a + (b + c)$

e.g. $2 + 3 + 5 =$

$(2 + 3) + 5 = 5 + 5 = 10$



$2 + (3 + 5) = 2 + 8 = 10$



Does it work for subtraction?

$9 – 6 – 3 =$

$(9 – 6) – 3 = 3 – 3 = 0$

$9 – (6 – 3) = 9 – 3 = 6$

Addition is associative but subtraction is not associative.

Possible task for students

Loretta has written the following number sentence

$$34 + 29 = 33 + 30$$

She did not have to add up the numbers to know this. Why?

Two students' responses to 34 + 29 = 33 + 30

One Year 6 student said, "Loretta just knows that they both add up to 63".

A Year 5 student said: "Loretta can do this because she did it in her head".

Neither student could explain why Loretta did not need to add the numbers in order to know that she was correct without using an explanation based on computation.

One student's response

One Year 5 student drew the following:

+1

$$34 + 29 = 33 + 30$$

–1

Referring to the 29, the student wrote:

"It increases by 1 to give 30, so 34 has to decrease by 1 to give 33".

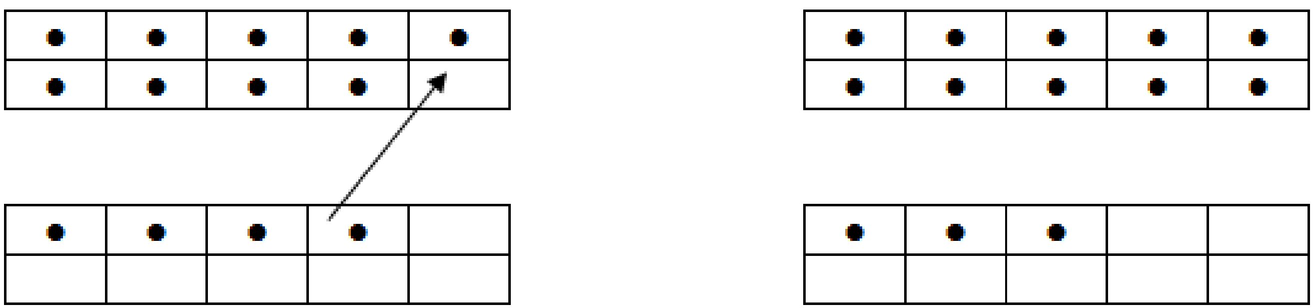
Features of relational thinking

- the focus is on the sentence, viewed as a whole
- the equals symbol stands for equivalence or balance
- relational thinking depends on being able to refrain from calculation (i.e. keep the sentence open)
- comparing pairs of known numbers (either side of the equals sign) to find the missing value.
- the strategies depend on the nature of the numbers and the operations involved

How could this thinking be modelled?

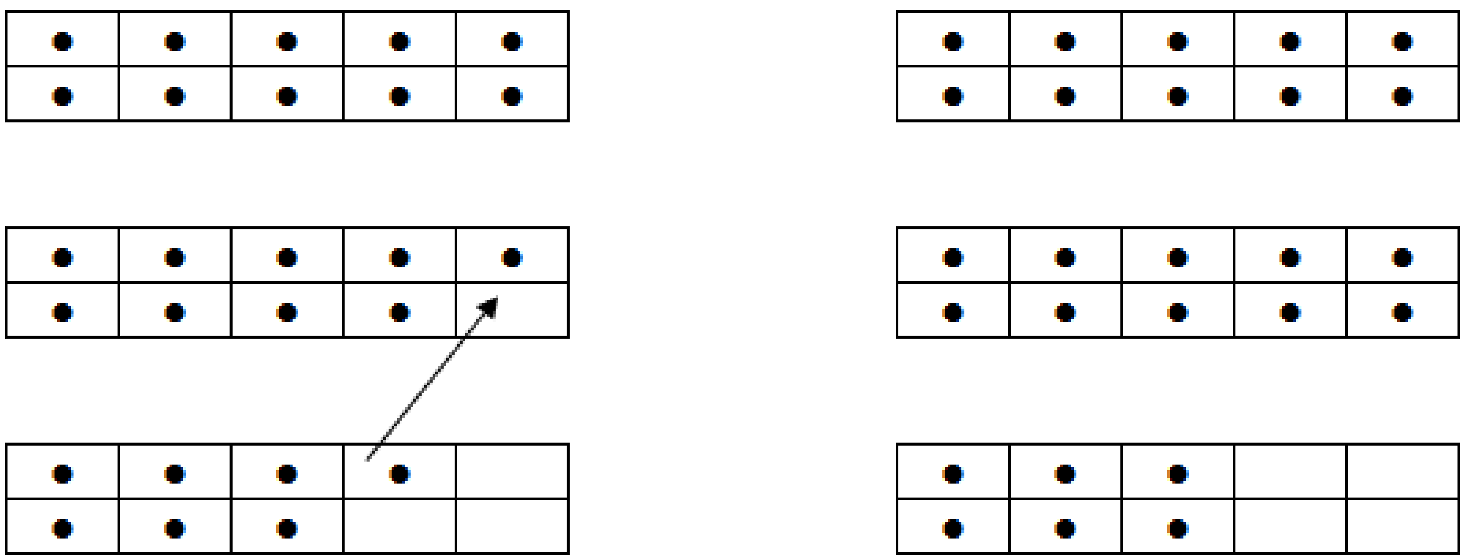
Tens Frames as a powerful model

9 + 4 is the same as 10 + 3



How could this thinking be modelled?

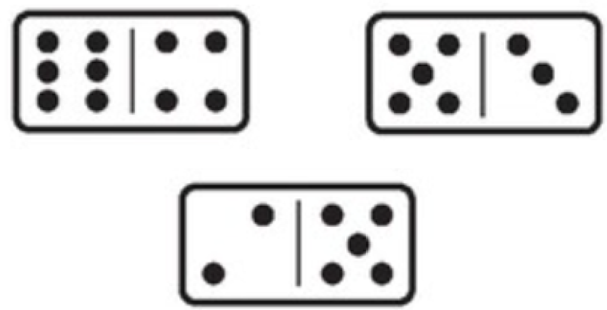
19 + 7 is the same as 20 + 6



Fact families using Dominoes

Choose one domino and write all the related facts shown on the domino you have chosen

See Dominoes at https://nrich.maths.org/1200



- What is the purpose of this task?
- What do students need to understand?

Fact Families using Dominoes

6 + 4 = 10

4 + 6 = 10

10 – 6 = 4

10 – 4 = 6

10 = 6 + 4

10 = 4 + 6

4 = 10 – 6

6 = 10 – 4

5 + 3 = 8

3 + 5 = 8

8 – 5 = 3

8 – 3 = 5

8 = 5 + 3

8 = 3 + 5

3 = 8 – 5

5 = 8 – 3

2 + 5 = 7

5 + 2 = 7

7 – 2 = 5

7 – 5 = 2

7 = 2 + 5

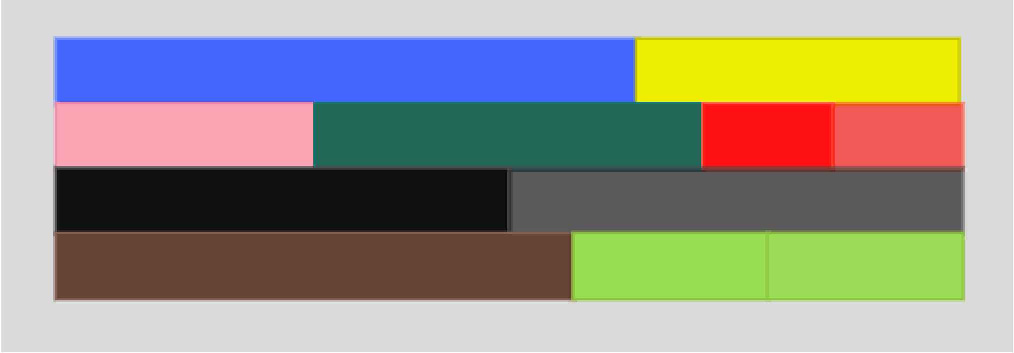
7 = 5 + 2

5 = 7 -2

2 = 7 - 5

Cuisenaire rods

See https://nrich.maths.org/4348
(online Cuisenaire environment)



If the blue rod is 9 and yellow rod is 5, what other number sentences could you write?

Find the missing numbers.

	My number sentence	Left side	Right side	Equal?
5 + □ = 3 + 4	5 + 2 = 3 + 4	5 + 2 = 7	3 + 4 = 7	Yes
9 + □ = 6 + 8				
□ + 6 = 9 + 9				
□ + 11 = 13 + 6				
7 + 21 = □ + 11				
17 + 15 = 8 + □				

http://smartvic.com/teacher/mdc/structure/ST40004P.html

Keeping the sum the same

Find the missing number in each of these number sentences by keeping the **sum** the same.

19 + 37 = ☐ + 41
19 + 267 = ☐ + 259
63 + 18 = 68 + ☐
58 + 23 = 54 + ☐
88 + ☐ = 97 + 265
☐ + 137 = 149 + 56

Keeping the difference the same

Find the missing number in each of these number sentences by keeping the **difference** the same.

18 – 11 = ☐ – 9
72 – 15 = ☐ – 25
23 – 14 = 28 – ☐
296 – 118 = 300 – ☐
113 – ☐ = 118 – 72
☐ – 137 = 294 – 150

The commutative property: Multiplication

The commutative property applies to addition and multiplication. Written symbolically the commutative properties say that no matter what numbers *a* and *b* are used:

$$a \times b = b \times a \text{ (commutative property for multiplication)}$$

- This means that for addition and multiplication, it doesn't matter which of the two numbers you start off with, and which number you use as the addend or the multiplicand (i.e. the number that gets added on or multiplied).
- Multiplication is commutative **but** division is NOT commutative

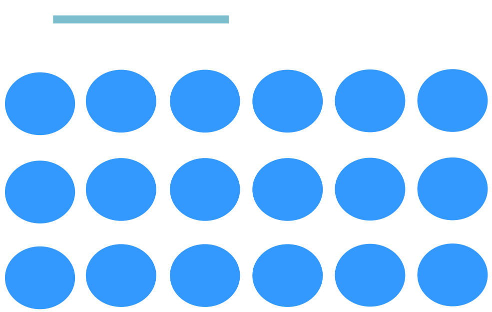
e.g. $2 \times 8 = 16$ and $8 \times 2 = 16$ **but**

$16 \div 8 = 2$ and $8 \div 16 = {}^{1}/_{2}$

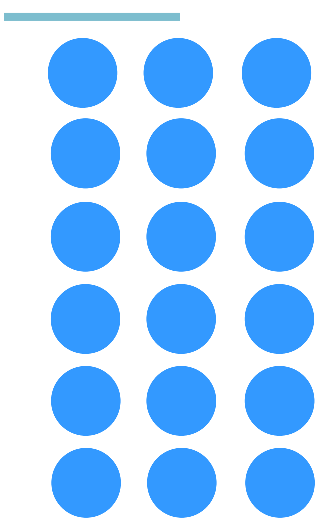
Multiplication is commutative

This halves the number of multiplication facts that children have to learn.

3 x 6 is the same as 6 x 3



3 rows of 6



6 rows of 3

The Associative Property

Multiplication is associative e.g. $3 \times (2 \times 60) = (3 \times 2) \times 60$

In general, $a \times (b \times c) = (a \times b) \times c$

Can use with commutativity to make mental calculation easier:

$5 \times 18 \times 2 = 5 \times 2 \times 18 = 10 \times 18 = 180$

Division is NOT associative

e.g. $16 \div (8 \div 4) = 16 \div 2 = 8$

BUT $(16 \div 8) \div 4 = 2 \div 4 = {}^{1}/_{2}$

The Distributive Property

The distributive property shows how multiplication works with addition. Written in symbols, the distributive property says that for three numbers *a* , *b*, and *c*,

$a \times (b + c) = a \times b + a \times c$

e.g. $10 \times (3 + 2) = 10 \times 3 + 10 \times 2$

- useful for mental computation
- the basis of all formal multiplication algorithms and
- used extensively in algebra for factorisation.

Fact Families

Write as many different multiplication or division number sentences as you can using only numbers from the set of numbers:

3, 4, 5, 12, 15, 20 (e.g. $3 \times 4 = 12$ and $12 \div 4 = 3$).

Fact Families

Just using numbers 3, 4 and 12 we can write the following:

$3 \times 4 = 12$

$4 \times 3 = 12$

$12 \div 4 = 3$

$12 \div 3 = 4$

$12 = 3 \times 4$

$12 = 4 \times 3$

$3 = 12 \div 4$

$4 = 12 \div 3$

Algebraic Reasoning

$$36 \times 25 = 9 \times \square$$

$$48 \times 2.5 = \square \times 10$$

$$\frac{2}{3} \times \square = 18$$

- Think about how you would solve these.
- What strategies would you use?
- What do students need to know to be able to solve this task?
- What might they find difficult?

$$5 \times \underset{\text{Box A}}{\square} = 10 \times \underset{\text{Box B}}{\square}$$

$$5 \times \underset{\text{Box A}}{\square} = 10 \times \underset{\text{Box B}}{\square}$$

b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

Algebraic Thinking Questionnaire Results

Overall results from 600 Year 5 – Year 9 students

Part M

1. For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$36 \times 25 = 9 \times \square$

56% gave a correct response

$48 \times 2.5 = \square \times 10$

46% gave a correct response

$\frac{2}{3} \times \square = 18$

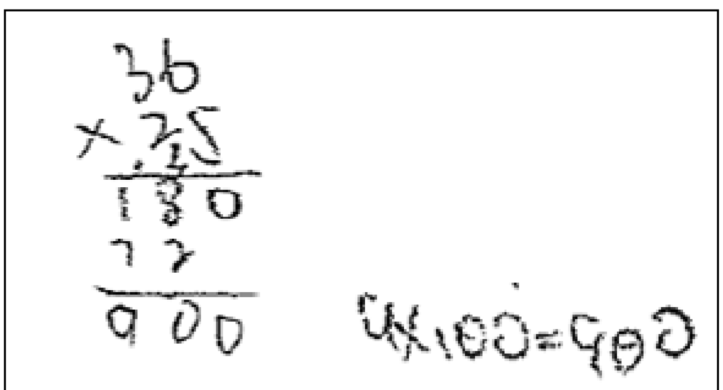
20% gave a correct response

$\frac{2}{5} \times \frac{\square}{\square} = 1$

22% gave a correct response

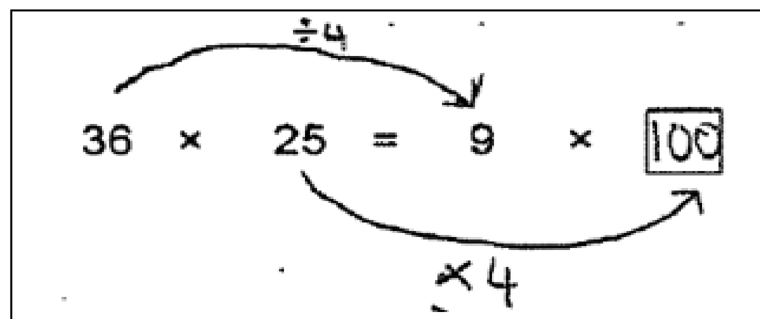
Algebraic Thinking Questionnaire: Task M1a

Student A used arithmetical calculations that demonstrate the *sameness-relational* understanding of the equals sign and ensured that the expressions on both sides of the equal sign were equivalent.



Student B demonstrates *substitutive-relational* understanding using arrows.

This student recognised that 36 divided by four is nine and that, in order to maintain the equality of the two expressions, multiplied 25 by four to get 100.



Algebraic Thinking Questionnaire Results

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$48 \times 2.5 = \square \times 10$

46% gave a correct response

$\frac{2}{3} \times \square = 18$

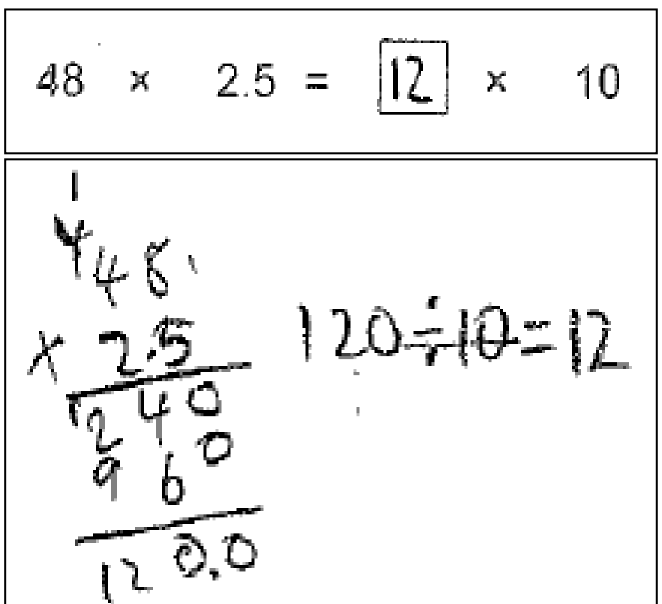
20% gave a correct response

$\frac{2}{5} \times \frac{\square}{\square} = 1$

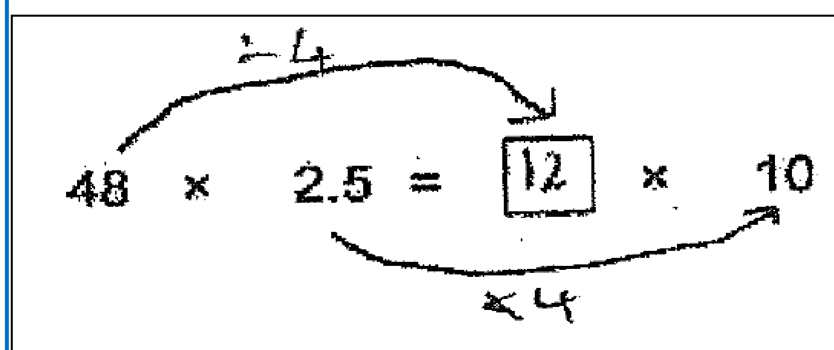
22% gave a correct response

Algebraic Thinking Questionnaire: Task M1b

Student C used arithmetical calculations that demonstrate the *sameness-relational* understanding of the equals sign and ensured that the expressions on both sides of the equal sign were equivalent.



Student D used relational thinking that demonstrates the *substitutive-relational* understanding of the equals sign. This student recognised that 2.5 multiplied by four is 10 and that, in order to maintain the equality of the two expressions, divided 48 by four to get 12.



What would you write?

$5 \times \underset{\text{Box A}}{\boxed{}} = 10 \times \underset{\text{Box B}}{\boxed{}}$

$5 \times \underset{\text{Box A}}{\boxed{}} = 10 \times \underset{\text{Box B}}{\boxed{}}$

b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

What would you expect your students to write?

Algebraic Thinking Questionnaire: Task M2a

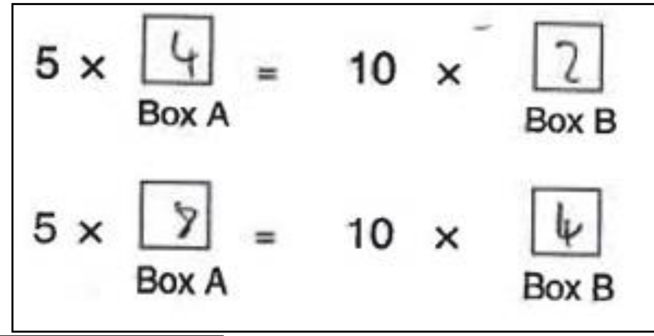
64% of all Years 5 – 9 students gave two correct pairs and 16% gave one correct pair but 20% either did not respond or gave incorrect answers

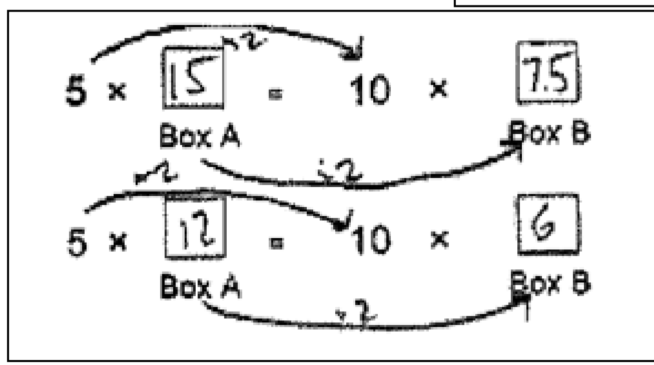
54% Year 5 gave 2 correct pairs (19%)

67% Year 6 gave 2 correct pairs (10%)

73% Year 8 (13%)

93% Year 9 (7%)





39% of all students correctly stated that the number in Box A is two times the number in Box B or that the number in Box B is one-half the number in Box A.

Connect, challenge, extend …

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Thank you

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